ADCC-GARCH(1,1)

Two asset return series $R_{i,t}$ (one for stocks and one for bonds) are specified in a GARCH(1,1) process (Ghalanos, 2022). The return-generating process is conceptualized as:

$$R_{i,t} = \mu_i + \varepsilon_{i,t}$$
$$\varepsilon_{i,t} \sim \mathrm{N}(0, s_{i,t}^2) \tag{1}$$

In the above equations, i is the subscript for stocks and bonds {i=Stock, Bond}. Subscript t denotes the day. $R_{i,t}$ is the daily raw log return of the stock and bond on time t which is a function of μ_i and $\varepsilon_{i,t}$. μi is the mean stock or bond log return and is assigned a value so that the return shocks $\varepsilon_{i,t}$ follow a zero-mean white noise process. The error term $\varepsilon_{i,t}$ follows a heteroskedastic normal distribution with mean 0 and conditional variance $s_{i,t}{}^2$ that is estimated in a univariate GARCH model. The residuals are thus assumed to be normally, independent and identically distributed.

The GARCH(1,1) model is a widely used GARCH specification in volatility modelling in which the (1,1) in parentheses refer to the number of autoregressive lags (or ARCH terms) and the number of moving average terms (the GARCH terms), respectively (Engle, 2001). The threshold GJR-GARCH(1,1) models the conditional variances series of the returns shocks $\varepsilon_{i,t}$ as follows:

$$s_{i,t}^2 = \omega_i + \varphi_i s_{i,t-1}^2 + (\nu_i + \delta_i I_{i,t-1})\varepsilon_{i,t-1}^2$$
$$I_{i,t-1} = \begin{cases} 0, & \varepsilon_{i,t-1} > 0 \\ 1, & \varepsilon_{i,t-1} \leq 0 \end{cases} \tag{2}$$

In the above equation, ω_i is the estimated long-run variance, $I_{i,t-1}$ is an indicator (dummy variable) with the value of one when the lagged shock is negative and a value of zero otherwise. This indicator makes that the effect of negative shocks on the volatility, as measured by $(\nu_i + \delta_i)$, is economically higher than the positives shocks that are only captured by ν_i (Alomari et al., 2021). δ_i is thus a parameter that accounts for the asymmetry in positive and negative shocks and measures the additional impact of lagged negative return shocks. ν_i measures the impact of a lagged return shock on the conditional variance. φ_i measures the impact of the lagged conditional variance on the conditional variance at time t. The persistence of a GARCH model measures how fast volatility decays after a shock and is given by $\upsilon + \delta\kappa + \varphi$ with κ being the probability that the standardized residual z_t is below zero $\mathrm{P}[z_t < 0]$. The asymmetric univariate GARCH model is necessary as it calculates conditional variances at each time t of the sample period, which are later used to calculate the standardized residuals. Parameters ω_i, φ_i, ν_i and δ_i are derived by applying the maximum-likelihood-estimation method (Ghalanos, 2022).

The ADCC model (Capiello et al., 2006) requires standardized residual series as an input to estimate dynamic correlation coefficients. Now that the residuals of the returns series $\varepsilon_{i,t}$ and their variance s^2 have been estimated, the conditional correlation coefficients can be derived by first normalizing the residual series as follows:

$$z_{i,t} = \frac{\varepsilon_{i,t}}{s_{i,t}} \text{ with } z_{i,t} \sim \mathrm{N}(0, q_{i,t}) \tag{3}$$

This standardized return shock series is normally distributed with mean zero and conditional variance q_t, which has an expected value of 1. Hereafter, the conditional covariance for the two normalized residual series of stock and bond returns is specified as:

$$\begin{gathered} q_{SB,t} = (1 - a - b - \kappa)\overline{\rho_{SB}} + a\, z_{S,t-1} z_{B,t-1} + b q_{SB,t-1} + \kappa \lambda_{S,t-1} \lambda_{B,t-1} \\ \lambda_{i,t} = \max\,[0, -z_{i,t}] \end{gathered} \tag{4}$$

$z_{i,t}$ is the normalized residual. $q_{i,t}$ is the conditional variance of the normalized residual. $q_{SB,t}$ is the conditional covariance for the two normalized residual series of stock and bond returns.

$\overline{\rho_{SB}}$ is the unconditional correlation coefficient for the two return series over the sample given by:

$$\overline{\rho_{SB}} = \frac{cov(R_S, R_B)}{\sigma_{R_S} \sigma_{R_B}} \tag{5}$$

In equation above, the autoregressive character can be seen in the $t-1$ subscript of the three last terms, more specifically the conditional covariance at time t being dependent of the conditional covariance of time $t-1$. a, b, κ are parameters to be estimated by the maximum log-likelihood-method. Parameter a measures the joint effect of the standardized residuals for stocks and bonds, b measures the effect of the conditional covariance and κ measures the joint effect of asymmetry in the standardized residuals. $(1 - a - b - \kappa)$ measures the impact of the long run correlation over the sample. The covariance at time t is thus for some part adjusted by the unconditional correlation coefficient for the two return series over the sample.

The above estimated conditional covariance for the two normalized residual series of stock and bond returns, as well as the individual variances of the normalized residuals for stocks and bonds are now used to estimate the dcc-correlation coefficients.

$$\hat{\rho}_{SB,t} = \frac{q_{SB,t}}{\sqrt{q_{S,t}}\sqrt{q_{B,t}}} \tag{6}$$

Taking together equation 4 and 6, $\hat{\rho}_{SB,t}$ is the dynamic conditional correlation coefficient between stock and bond returns. This coefficient is calculated for each of the seven financial markets. the stock-bond return covariance on time t is for some part explained by the past covariance, and is each day adjusted by the product of standardized return shocks. As a result, the stock-bond return correlation on time t is for some part dependent on the previous correlation which is adjusted by the sign and magnitude of the standardized residuals and their conditional variance. When both shocks have the same sign (positive or negative) (and thus the product of the standardized return shocks is positive), the dynamic correlation on time t is constructed by considering the correlation at time $t-1$ and upwardly adjusting this $t-1$ correlation. The opposite applies when stocks and bond standardized return shocks have an opposite sign.

In conclusion, the ADCC-GARCH(1,1) model requires two daily return series $R_{i,t}$ over a certain sample period as input variables. After model implementation, one can retrieve a time series which contains the daily dynamic conditional correlation coefficients between the two return series over that same sample period. These conditional correlation coefficients have an autoregressive and moving average (both of order 1) character.

Hedge Ratio

The hedge ratio on day T is given by:

$$\beta_T = \frac{\hat{\rho}_{SB,T-1}\sqrt{q_{S,T-1}}\sqrt{q_{B,T-1}}}{\hat{\sigma}^2_{B,T-1}}$$

$q_{S,T}$ and $q_{B,T}$ are the conditional variances of the standardized residual series $z_{i,T}$ of stock and bond returns. $\hat{\rho}_{SB,T}$ is the dynamic conditional correlation, $\hat{\sigma}^2_{B,T-1}$ is the conditional variance of the government bond returns that was independently generated through a GJR GARCH(1,1) process. As $q_{S,T}$ and $q_{B,T}$ and $\hat{\sigma}^2_{B,T-1}$ are positive, one can easily notice that hedging by using negatively correlated assets results in negative hedging ratios. In a long-only scenario, one should thus only consider negative hedging ratios. This gives:

$$\beta_T = 0 \; if \; \beta_T > 0$$

Hedging Effectiveness

Construct an active portfolio of which the weight distribution between stocks and bonds on time T is given by:

$$\omega_{Stock,T} = \frac{1}{1-\beta_T} \; with \; \beta_T \leq 0$$

$$\omega_{Bond,T} = 1 - \omega_{Stock,T}$$

These weight distributions are used to construct the active portfolio. This portfolio is compared to the passive (60/40) portfolio in terms of hedging effectiveness given by:

$$H_T = 1 - \frac{\hat{\sigma}^2_{Active,T}}{\hat{\sigma}^2_{Passive,T}}$$

$\hat{\sigma}^2_{Active}$ and $\hat{\sigma}^2_{Passive}$ are the univariate conditional variances on time T of respectively the active and passive portfolio, estimated by the GJR-GARCH(1,1) model.